Bi-Global Linear Stability Analysis

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1 Introduction

The linear stability analysis assumes the knowledge of a so-called basic flow, the latter being the flow the stability of which is studied. In the linearized Navier-Stokes equations, the quantities related to the basic flow (velocity, pressure, temperature ...) act as coefficients, whereas the same quantities related to the fluctuation act as the unknowns.

The classical stability analyses concern shear basic flows (Kelvin-Helmholtz, boundary layer, jet flows ...) which implies that the mean flow velocities mainly depend on one space variable only. The basic flow is inhomogeneous with respect to one space variable only. In that case, the perturbation may be searched as a normal mode which means that the wave-like structure is modulated by an amplitude function which depends on the space coordinate describing the inhomogeneous direction of the basic flow (the shear direction). This form finally leads to an ordinary differential equation (ODE) written for the amplitude function. In the case of an incompressible viscous flow this ODE corresponds to the famous Orr-Sommerfeld equation.

However in some cases the basic flow exhibits not only one direction of inhomogeneity but two, this means the basic flow is inhomogeneous in a plane, let us write (X, Y) a coordinate system for this plane. In that case, the fluctuating quantities q_{fluc} are searched with an amplitude function which depends on these two coordinates :

$$q_{\text{fluc}} = \hat{q}(X, Y) \exp\big(i(kZ - \omega t)\big)$$

with k a wave number and ω usually a complex number, the real part representing a circular frequency and the imaginary part the temporal growth rate of the perturbation. For this specific form of the fluctuation the linearized Navier-Stokes equations lead to partial differential equations system (PDE) instead of the previous ODE. The eigenvalue problem is then posed in terms of a PDE system and is called a "bi-global" stability problem (in the sense of a two directional eigenvalue problem).

This PDE obviously requires more computational power, mainly in terms of the memory (RAM), than the classical ODE eigenvalue problem. However thanks to the improvement of the computers capacities, the bi-global stability approach is now applied to different configurations by different teams in Europe (and elsewhere of course). To solve the PDE eigenvalue problem, a shooting method can be applied at least in some cases or the spectrum can be calculated. In this second method, for computing time reasons, only a part of the spectrum is calculated thanks to the so-called Arnoldi algorithm.

It may be important to distinguish two type of bi-global stability applications depending on the boundary conditions.

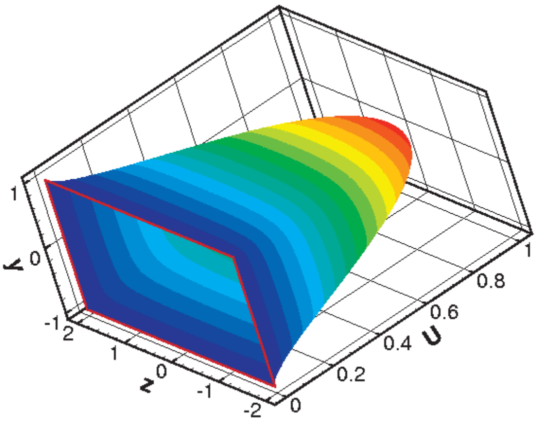

Figure 1: Basic flow axial velocity for the Poiseuille flow through a rectangular duct

2 Physical boundary conditions

The bi-global approach has been first applied to a Poiseuille flow through a rectangular section see [1] and has been then extensively used by Theofilis for different applications, see [2] for a review. The case of the confined Poiseuille flow has been more recently re-analyzed within the PhD work of C. Robitaillié-Montané [3]. Figure 1 shows the basic axial velocity which depends on two spatial directions here denoted by y and z. The flow is bounded by solid walls which leads to physical boundary conditions (no-slip conditions written for the considered viscous flow). The figure corresponds to an aspect ratio equal to 2. An example of results is given in figure 2 in case of an aspect ratio of 5. The iso values of the norm of the axial fluctuation velocity function is plotted as function of y and z. Close to the middle $(z = 0)$ the shape of the function $y \mapsto u(y, z)$ is close to the usual shape obtained for the 1D problem (corresponding to an infinite aspect ratio).

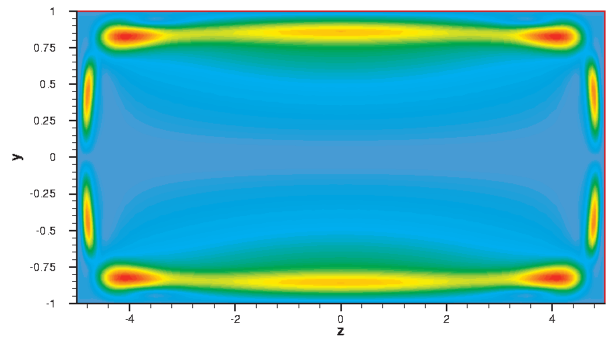

Figure 2: Iso axial fluctuating velocity function for the Poiseuille flow through a rectangular duct.

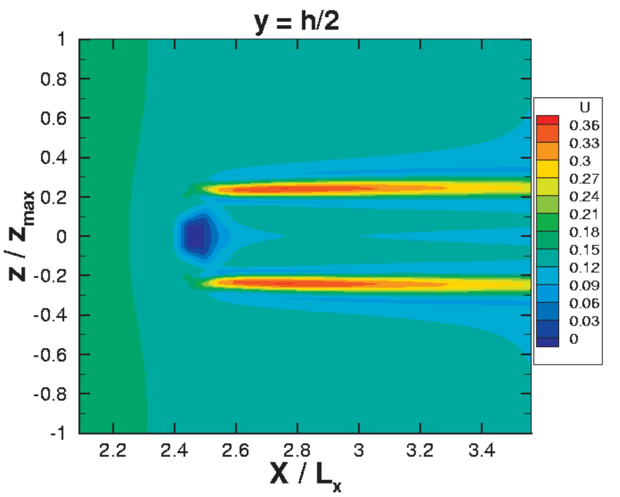

Figure 3: Iso value of the axial basic flow velocity at the distance $h/2$ from the plate, with h the height of the roughness element.

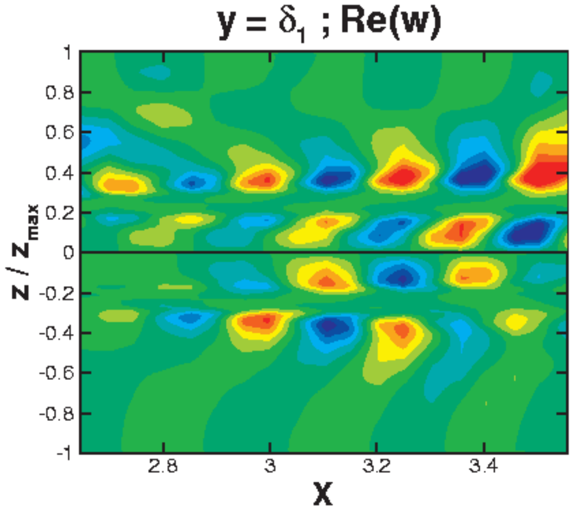

Figure 4: Real part of the spanwise fluctuating flow velocity at the distance δ_1 from the plate, with δ_1 the displacement thickness of the local boundary layer. Upper part ($z > 0$) comes from the bi-global stability, whereas the lower part is the DNS result obtained in Stuttgart.

Another example of bi-global stability analysis with physical boundary conditions has been recently initiated by Estelle Piot [4] in collaboration with Ulrich Rist. A roughness elements array is periodically placed on the surface of a flat plate at a given distance from the leading edge. A basic flow can be computed around a single roughness element with periodic boundary conditions when the boundaries are placed exactly between two roughness elements. This computation has been done in Stuttgart by A. Wőrner and U. Rist. An example of result is provided in figure 3 which shows the axial basic flow velocity at the distance $h/2$ from the wall where h is the height of the roughness element. As it can be easily observed, the roughness elements array induces large modifications in comparison with the flow without roughness elements. In particular it appears vortices-like structures which start just downstream the roughness element on each side of it. Finally at some distance downstream the roughness element a steady flow slightly evolving with respect to x (weakly non parallel) is observed which is strongly inhomogeneous in the (y, z) directions. A bi-global linear stability approach has been then undertaken by E. Piot in order to analyse the stability properties of this basic flow. Each fluctuating quantity is searched under the following mathematical form :

$$q_{\text{fluc}} = \hat{q}(y, z) \exp\big(i(kx - \omega t)\big)$$

where ω is a real number and k a complex number, its real part representing an axial wave number and its imaginary part a spatial growth rate. Collecting the different results corresponding to different x values, a theoretical eigenmode is calculated and can be compared to the unsteady results obtained in Stuttgart thanks to their DNS approach. An example of comparison is given in figure 4. The real part of the spanwise fluctuation is plotted at the distance δ_1 from the wall, where δ_1 is the displacement thickness (at the roughness element position). The upper part (positive z values) of the figure has been obtained with the bi-global stability approach whereas the lower part (negative z values) has been obtained by DNS in Stuttgart. A very good agreement is obtained, except close to the roughness where the large non parallel effects can strongly modify the stability results and close to the most downstream locations where the DNS may be not fully converged.

3 Non-physical boundary conditions

The bi-global stability analysis may be also used for configurations including a non-physical boundary condition, usually an in-flow condition at an upstream position and/or an out-flow condition at a downstream position. A typical example is given for the stability analysis of a recirculation zone occurring in a two-dimensional boundary layer. This typical configuration has been studied for instance by V. Theofilis, [2], it is also currently investigated in ENSAM (Paris) by J. Ch. Robinet and coworkers, in Marseille by U. Ehrenstein and in KTH by E. Åkervik.

An other typical configuration is the flow over a rectangular cavity placed on a flat plate. The stability analysis is then performed in relation to aeroacoustics predictions. Such configurations are investigated for instance in KTH by D. Henningson and coworkers and in ONERA (Paris) by D. Sipp.

It is also important to mention the linear stability analysis of the attachment line flow along the leading edge of an infinite swept wing. This particular basic flow can be analyzed thanks to the standard ODE eigenvalue approach using the so-called Görtler-Hämmerlin approach. However the bi-global stability approach may also be used, see [5]. It is then proved that the Görtler-Hämmerlin is only one of the possible amplified eigenmode (even if it remains the most amplified one). The same configuration has also been analyzed thanks to the bi-global stability approach in case of compressible swept attachment-line boundary layer, see [3]. In this case, the bi-global stability approach is the only possible approach for the calculation of the eigenmode. It is then proved that the Görtler-Hämmerlin mode and the other modes obtained in incompressible flow are continuously transformed into modes satisfying the linearized equations written for a compressible flow, the first one remaining the most amplified one. However for supersonic flows, other modes appear which are more amplified than the previous ones.

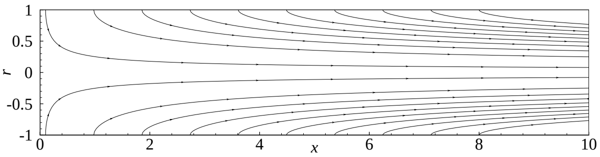

Figure 5: Velocity field and streamlines of the flow induced by lateral wall injection, the front wall located at $x = 0$ being a solid wall through which there is no injection.

Another configuration is also studied at ONERA with the help of the bi-global stability approach. This study is performed in order to determine the origin of the thrust oscillations which occur in the two boosters of the European launcher Ariane 5. Despite the severe conditions (high pressure, high temperature, two phase flow, combustion ...) the flow inside the booster can be easily modelled by a simple flow inside a semi-infinite pipe where the flow is injected inside the pipe through the lateral wall at a uniform and constant velocity. Analytical relationships are available for this particular flow, they have been obtained separately by Taylor and Culick, the two components (in the usual cylindrical coordinate system) of the dimensionless axisymmetric basic flow are :

$$\begin{cases} U_x = \pi x \cos\left(\dfrac{\pi r^2}{2}\right) \\ U_r = -\dfrac{1}{r}\sin\left(\dfrac{\pi r^2}{2}\right) \end{cases}$$

The flow is strongly inhomogeneous with respect to the (x, r) coordinates. Some streamlines are plotted in figure 5. This flow has been extensively studied within the PhD work of F. Chedevergne, see [6] for instance. The bi-global stability analysis of this flow imposes to introduce a non-physical boundary at some specific position x and ad-hoc boundary conditions must be written as out-flow conditions. For this particular flow, it has been proved that a condition such as an extrapolation and a zero normal derivative for the fluctuation provide similar results and that it does not affect the eigenfunction except in a region very close to this artificial boundary.

The bi-global linear stability approach of this specific flow leads to different eigenvalues and thus to different possible frequencies. The frequency has a fixed non-dimensional value but a variable dimensional value thanks to the increase of the fluid region : the fluid region radius increases with respect to time during the firing because the solid propellant which is actually the "lateral wall" burns continuously during the firing.

Finally the bi-global stability modes are associated to dimensional frequencies which evolve during the firing, their evolution can be plotted in a time-frequency diagram as it is shown by solid lines in figure 6. In addition experimental results obtained for a reduced scale motor tested at ONERA by M. Prévost, see [6], are also plotted. It can be deduced that the most amplified measured frequencies are in excellent agreement with the theoretical predictions coming from the use of the linear bi-global stability analysis.

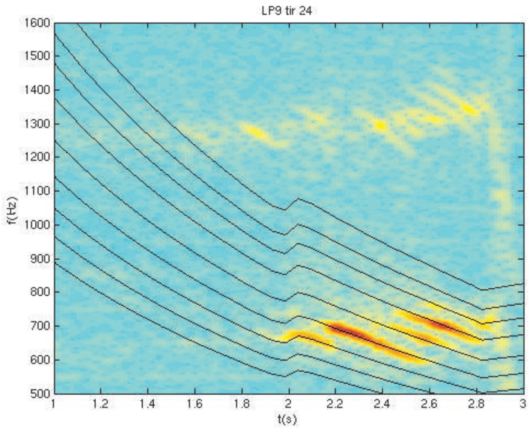

Figure 6: Amplified frequencies in the diagram time-frequency. The solid lines correspond to the dimensional values of the bi-global stability eigenmodes evolving with respect to the time whereas the colors correspond to the measurements performed at ONERA by M. Prévost., see [6].

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