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Exponential-algebraic transition in the near-field of low supersonic jets

Christophe Millet *,[1], Grégoire Casalis

ONERA-Toulouse, Aerodynamics and Energetics Modeling Department, 2, avenue Édouard Belin, BP 4025, 31055 Toulouse cedex 4, France

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Abstract

The paper is concerned with the transition from exponential to algebraic cross-stream decay of instability waves in supersonic axisymmetric cold jets. The cross-stream structure of these waves is analysed for phase velocities close to the sound speed when the streamwise inhomogeneities of the mean flow are characterized by a small parameter. It is shown how algebraic decays are completely compatible with the features of the near-field, and a selection criterion for the occurence of such a behaviour is given. Near-field pressure fluctuations are then determined as functions of azimuthal properties of instability waves and control parameters. In some cases, where the dominant contribution to the sound generation is due to the axisymmetric Kelvin–Helmholtz instability mode, algebraic decays are not confined in the core region of the jet and the acoustic wavelength becomes larger than the envelope scale.
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1. Introduction

For perfectly expanded jets, many measurements have shown that large-scale coherent structures are dominant noise producers in supersonic jets (see Seiner and Ponton [1], Troutt and Mclaughlin [2], Lau et al. [3,4]). These results have been confirmed by theoretical studies (Mankbadi and Liu [5], Tam and Burton [6,7]) which revealed that only contributions that arise from these structures can be retained in the aerodynamic sound integral. The shape distribution of the coherent structures responds to the local profiles of the mean shear flows and may be calculated by using the hydrodynamic stability theory. The fine-grained turbulence plays an indirect but crucial role in that it controls the development of the coherent structures and consequently its emitted sound.

An approximate picture of the physical mechanism by which the large turbulence structures generate sound is to regard the instability wave as a wavy wall (Tam [8]). This analogy suggests that the direction of the most intense noise radiation from highly supersonic jets can be estimated by using the Mach angle relation based on the speed of the most amplified instability wave (Tam et al. [9]). However, most of previous works are confined to highly supersonic jets, for which the most amplified instability wave has a supersonic phase velocity (relative to ambient sound speed) and do not consider the subsonic–supersonic transition. In some cases, such a transition gives rise to acoustic radiation: the cross-stream behaviour of pressure fluctuations becomes dispersive and the cross-stream decay of the amplitude changes from exponential to algebraic. The present study is further motivated by the fact that, this exponential-algebraic transition may depend strongly on the azimuthal properties of

* Corresponding author.
E-mail address: Christophe.Millet@cea.fr (C. Millet).
[1] Present adress: Geophysics Laboratory, CEA, BP 91680, Bruyères-le-Châtel, France.

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instability waves and control parameters. Accordingly, there is no reason to consider only the most amplified instability wave for the determination of the dominant noise source for low supersonic jets.

The primary goal of this work is to examine the phenomenon of exponential-algebraic transition of pressure near-field for low-supersonic jet flows by resorting to the linear stability theory. Recently, numerical simulations of supersonic jets and their sound fields have shown that linear stability can be used to estimate near-field sound pressure levels (Freund et al. [10], Mitchell et al. [11]). In Section 2 we study the decay of near-field fluctuations in a convectively unstable axisymmetric jet. We restrict the discussion to the Kelvin–Helmholtz instability. It is easy to write down a general integral solution to the problem outside the jet. But there is no universal way in which the exponential-algebraic transition is achieved, and indeed much of the emphasis here is to show that the transition depends on the azimuthal properties of instability waves. In this paper, it is shown that the exponential-algebraic transition is a sufficient condition for acoustic radiation in the far-field. Physically, when such a transition arises, the disturbances associated with the flow-instability process extend from the jet all the way to the far field. The problem is global in nature. In Section 3 the numerical formulation used to solve the governing stability equations is described. Here, a spectral collocation discretization through a multiple domain technique has been used for the calculations. This approach, which has been developed by Khorrami et al. [12], Malik [13] among others, allows us to include the viscosity terms when it is necessary. Numerical results are then presented in Section 4 for cold supersonic jets.

2. Near field structure of instability waves

2.1. Basic formulation

The complex fluctuations $\phi = (\mathbf{u}, p)$ around a given mean jet flow $\bar{\phi} = (\bar{\mathbf{u}}, \bar{p})$ are assumed to be governed by linearized inviscid, compressible equations of motion. Here the functions $\mathbf{u}$ and p are the velocity and the pressure, respectively. We will use r_0, the radius of the jet at the nozzle exit, u_0 the jet exit velocity and ρ_0 the jet exit density as length, velocity and density scales of the problem. The time and pressure scales are given by r_0/u_0 and $\rho_0 u_0^2$, respectively. In the sequel, we shall assume that the mean pressure is constant in the domain of motion Ω and the product of the mean velocity by the mean density belongs to the class consisting of solenoidal vectors (that is, $\bar{\rho}(\nabla \cdot \bar{\mathbf{u}}) + \bar{\mathbf{u}} \cdot \nabla \bar{\rho} = 0$) vanishing far away from the center of the jet flow. Thus, in dimensionless form, the perturbation $(\mathbf{u}, p)$ satisfies the following equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\bar{\mathbf{u}} = -\frac{1}{\bar{\rho}} \nabla p, \tag{2.1a}$$

$$\frac{\partial p}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla p) = -\frac{1}{M^2}(\nabla \cdot \mathbf{u}), \tag{2.1b}$$

in the class consisting of vectors $(\mathbf{u}, p)$ satisfying a radiation condition or boundedness condition outside the jet. Here the jet exit Mach number $M = 1/\sqrt{\gamma \bar{p}}$, γ beeing the ratio of specific heats, is a control parameter.

The non-turbulent fluid outside the jet is assumed to be inviscid so that the distribution of fluctuations is a solution of the above equations outside the jet. In this paper, the discussion is restricted to instabilities arising from purely inviscid mechanisms as the Kelvin–Helmholtz instability waves. It follows that Eqs. (2.1a), (2.1b) can be used in the whole of the domain Ω with the exception of regions where the inviscid approximation fails to represent the cross-stream behaviour of instability waves. For incompressible shear flows, Le Dizès et al. [14] have shown that this phenomenon occurs in large viscous regions when the instability waves are damped. From a numerical point of view, weakly viscous effects may be taken into account to compute the cross-stream behaviour of fluctuations in these regions. As the governing equations can be solved in a close form outside the jet, another possibility is to use the contour deformation method, as described by Tam and Morris [15].

In the present study, we are interested in the cross-stream structure of solutions to the above problem for a weakly non-parallel axisymmetric jet. Thus, the properties of the mean flow are assumed to be functions of a slow space variable $X = \varepsilon x$, where ε is a small parameter characterizing the streamwise inhomogeneities of the medium and x denotes the streamwise direction. With respect to a cylindrical coordinate system (x, r, θ) centred at the nozzle exit, the mean velocity may be represented analytically in the form

$$\bar{\mathbf{u}} : (x, r) \mapsto \big(\bar{u}(X, r), \varepsilon \bar{v}(X, r), 0\big). \tag{2.2}$$

Now let us consider the near-field region around the jet. Outside the jet, the streamwise component $\bar{u}$ is identically zero and the mean radial velocity component $\bar{v}$ can be calculated by integrating the mean continuity equation. Following Tam and Burton [7], the numerical value of $r\bar{v}$ is regarded as a constant and given by $\bar{v}_\infty$. It follows that the dimensionless mean fluid density is constant and uniquely determined from the control parameters, such as the Mach number M or the temperature of the jet exit. It will be denoted by $\bar{\rho}_\infty$. Such assumptions imply that the linear operator associated with (2.1a), (2.1b) is homogeneous in space variable X. Homogeneity allows us to make extensive use of Fourier–Laplace transforms to reduce the problem (2.1a), (2.1b) to an ordinary differential equation outside the jet. Another point of view consists in using the WKBJ approach to describe the instabilities. Since the flow is slowly evolving on the streamwise scale X, it is legitimate to decompose all solutions into local plane waves at any streamwise station X. Note that the latter mathematical description is known to be valid only inside and in the immediate vicinity of the jet (a proof is given by Tam and Morris [15]).

2.2. Cross-stream structure of local plane waves

The starting point of the analysis is the problem defined by (2.1a), (2.1b) and (2.2) which describes both the instability waves behaviour and the acoustic near and far field, with some conditions for ϕ on the boundary of Ω. Here, we are interested in the manner in which the near-field decay changes from exponential to algebraic around a location X – and more generally we are interested in the cross-stream structure of solutions to (2.1a) and (2.1b) outside the jet, for $\varepsilon \to 0$.

Since the jet is assumed to be axisymmetric, the complex fluctuations can be Fourier-decomposed into azimuthal modes. Eqs. (2.1a), (2.1b) being invariant under arbitrary time translations, it is legitimate to seek solutions of the form

$$\phi(x, r, \theta, t) = \frac{1}{2\pi} \int_{L_\omega} \phi_n(x, r, \omega) \exp(\mathrm{i}n\theta - \mathrm{i}\omega t)\, \mathrm{d}\omega, \tag{2.3}$$

where n is an integer and ω is the frequency; the integration is performed along the path L_ω in the complex ω-plane as sketched in Fig. 1. Since azimuthal and temporal dependances are specified throughout the entire flow, one has to solve a problem for which n and ω are now parameters.

Let X and R be defined by $X = \delta_X(\varepsilon)x$ and $R = \delta_R(\varepsilon)r$, where δ_X and δ_R are order functions. Then by eliminating all other dependent variables in favour of u_n, the streamwise component of $\mathbf{u}_n$, one can find the equation to be solved outside the jet

$$\delta_R^2 \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{n^2}{R^2} \right) u_n + \left(k_0^2 + \delta_X^2 \frac{\partial^2}{\partial X^2} \right) u_n = F(\varepsilon \bar{v}_\infty) u_n, \tag{2.4}$$

which will be written as $\delta_R^2 L_0 u_n + L_1 u_n = F u_n$, where L_0 is the transverse part of the Laplace operator and k_0 is defined by $k_0^2 = \bar{\rho}_\infty M^2 \omega^2$. The full expression for F is not of first importance here; it is given in Appendix. It seems natural to introduce, for the purpose of the analysis, the one parameter family of local variables $r_\alpha = R\varepsilon^\alpha / \delta_R(\varepsilon)$, with $\alpha \geqslant 0$ and to consider a local asymptotic approximation of u_n in the vicinity of the jet. Here, L_0 does not depend on ε, while for any (sufficiently differentiable) ε-independent function g, the linear operator F is such that $Fg = \mathrm{O}(\varepsilon \delta_R^2)$. For slowly divergent free shear flows, δ_X may be a measure of nonparallel basic flow effects, for instance $\delta_X(\varepsilon) = \varepsilon$. Then, the degeneration of (2.4) in the local variable r_α only depends on the order of magnitude of $\|L_1 u_n\|$ for $\alpha = 0$. For $0 < \alpha < 1$, the family of local variables plays the role of a family of intermediate variables.

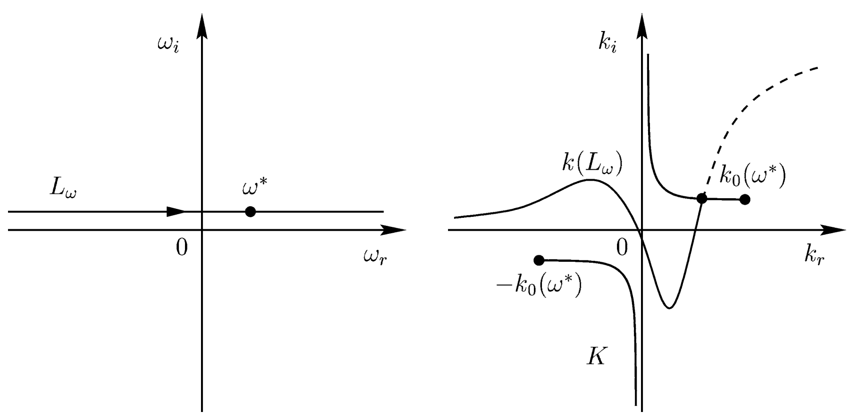

Fig. 1. Locus of spatial branch in complex k-plane as ω moves along L_ω contour in complex ω-plane for small ω_i. The branch cut K for the function λ join the critical points for $\omega = \omega^*$. Dashed line: analytic continuation into the second sheet of the Riemann surface (corresponding to solutions which become exponentially large as $r \to \infty$).

We will first consider the case $\alpha = 0$. As discussed in Section 2.1 the asymptotic response can be given by the standard form of the WKBJ approximation (see Bender and Orszag [16]) which describes instabilities of weakly non-parallel flows:

$$u_n \sim \sum_{j=0}^{q} \delta_j(\varepsilon) A_j(X) u_{nj}(r;\omega) \exp\left(\frac{\mathrm{i}}{\varepsilon} \int_0^X k(s;\omega)\,\mathrm{d}s \right), \tag{2.5}$$

where $\delta_0 = 1$, δ_j $(0 < j < q)$ are elementary gauge functions and $k = k_r + \mathrm{i}k_i$ denotes a local (complex) wavenumber originating in the upper half of the k-plane when the imaginary part of ω goes from a large positive value to zero. The leading-order amplitude $A_0(X)$ is determined at another order in ε, from matching conditions to the solution far away from the jet, and u_{n0} is a suitably normalized function which gives the cross-stream behaviour of pressure fluctuations in the vicinity of the jet. A cause of difficulties is the appearance of terms δ_j in the formal expansion of which the order of magnitude is not suggested or dictated by the differential equation or the boundary conditions. However, these functions can be determined by using the intermediate matching principle, as described by Tam and Burton [6,7]. Here, the origin $X = 0$ is the axial location of the source on the jet centreline.

By introducing the expansion (2.5) for u_n into Eq. (2.4) with $\alpha = 0$, one finds an equation satisfied by u_{n0} outside the jet,

$$L_0 u_{n0} - \lambda^2 u_{n0} = 0, \tag{2.6}$$

where λ is given by the irreducible polynomial

$$P(\lambda, k) = \lambda^2 + \left(k_0^2 - k^2\right) = 0, \tag{2.7}$$

which defines an algebraic function $\lambda(k;\omega)$ in a domain of the complex k-plane. Using the new variable $z = \mathrm{i}\lambda r$, Eq. (2.6) reduces to the Euler–Bessel equation in the z-complex domain. Two linearly independent solutions of (2.6) are then given by the n-th-order Hankel functions of the first and second kind, $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$, respectively. The passage to the general case of an arbitrary positive value for α causes no difficulty: one must consider the transformation $r = r_\alpha/\varepsilon^\alpha$ in the definition of z. In the following, we exclude situations where the leading-order of Eq. (2.4) reduces to $L_0 u_{n0} = 0$ by reducing the analysis to the plane of the complex variable k with the exception of neighbourhoods of branch points $\pm k_0$.

Let us consider now the algebraic function determined by the polynomial (2.7) in the domain $\mathcal{E}$ obtained by removing the singular points from the complete k-plane, that is, the points $\pm k_0 = \pm\sqrt{\bar\rho_\infty}\,\omega M$ and ∞. Then at every point $k \in \mathcal{E}$ this equation has two distinct finite roots λ_1 and $\lambda_2 = -\lambda_1$, associated with regular branches. In the strictly parallel case, the real part of one branch, let us say $\lambda_1(k;\omega)$, can be taken positive, so that only the n-th-order Hankel function of the first kind, $H_n^{(1)}$, satisfies the boundedness condition on the boundary of Ω. Such a choice will be highlighted in Section 2.4. It implies branch cuts K in the k-plane as shown in Fig. 1 and such that

$$-\pi/2 \leqslant \arg\lambda < \pi/2. \tag{2.8}$$

Let us note in connexion with this, that the singular points $\pm k_0$ go to infinity when the real part ω_r of the frequency ω moves from zero along the contour L_ω. Then, we arrive at the result, that it may be possible to find at least one point k^* in the k-plane where a spatial branch meets a branch cut as shown in Fig. 1. Note also that it is clearly impossible to find such a point if the shear layer thickness is zero since the model would fail to satisfy causality or if the Mach number M is zero.

For weakly non-parallel flows, the local wavenumber $k(X;\omega)$ traces a path Λ in the k-plane from the point $k(0;\omega)$ which passes through the real axis, as shown in Fig. 2. One has to remember that, as the wave propagates downstream the local growth rate $-k_i(X;\omega)$ reduces. This is because the flow slowly diverges so that the transverse velocity gradient is gradually reduced. This path and the singular points continuously change as ω moves along L_ω, or with the continuous change of control parameters. More detailed investigation shows that only singular points of λ, that is the points not belonging to the domain $\mathcal{E}$, can act as an obstacle to the deformation of the path Λ. Hence it is natural to expect, that the character of the permutation of the functions λ_1 and λ_2, is determined by the distribution of the singular points on the sphere of the complex variable k. Therefore, we arrive at the following criterion for exponential-algebraic transition: *if at least one spatial branch $k(X; L_\omega)$ meets the branch cut K in the complex k-plane, then the cross-stream decay of the amplitude of pressure fluctuations is algebraic in a subdomain of Ω.*

This result follows by the permutation of branches, using the branch cuts K as defined above. For every intersection point k^*, the real part of λ is zero and $|k_r^*| < k_0$). Then, taking the intermediate limit of the Hankel function, r_α beeing fixed, we find

$$\lim_{r_\alpha} \left| H_n^{(1)}(\mathrm{i}\lambda r) \right| \sim \frac{|\exp(-\lambda r)|}{|\sqrt{\mathrm{i}\lambda r}|}. \tag{2.9}$$

It follows that the cross-stream decay of u_{n0} is algebraic for k^*. In the case where no intersection point is found, the cross-stream decay is exponential for every location X since the real part of λ_1 is positive and no travelling wave is generated in the

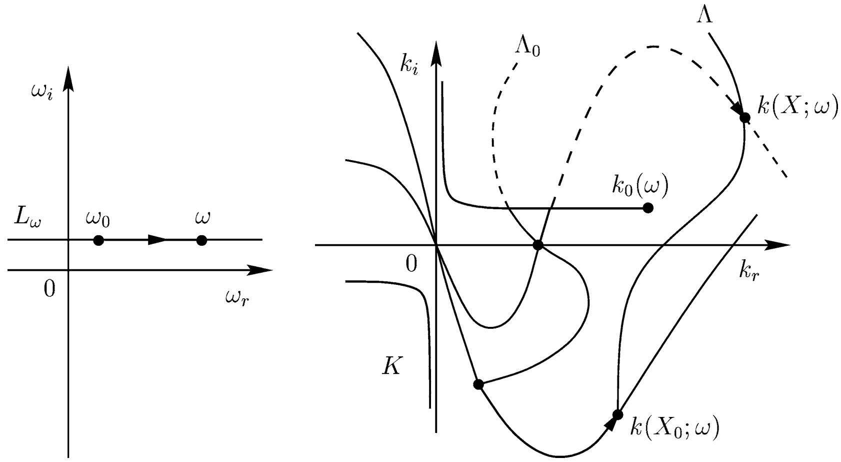

Fig. 2. Locus of spatial branch in complex k-plane as ω moves along L_ω contour, for two locations X_0 and $X > X_0$. In the displacement of the point ω along the L_ω contour from the point ω_0, the path Λ_0 is continuously deformed into the path Λ. The algebraic function $\lambda(k;\omega)$ continuously changes with the continuous change of the position of the point k on the Riemann surface.

cross-stream direction. In order to have a continuously change of the function λ with the continuous change of the wavenumber k, one has to define it not on the k-plane, where it is many valued, but on a 2-sheeted Riemann surface, to every point of which correspond one value of the function λ.

In gradually displacing the L_ω contour until the real axis, the imaginary parts of singular points $\pm k_0(\omega)$ go to zero and the parts of branch cuts not belonging to the imaginary axis coincide with the segment consisting in neutral waves ($k_i = 0$) with supersonic phase velocities ($|k_r| < k_0$). The length of this segment is an increasing function of the Mach number M, and is zero in the incompressible flow assumption. Then the criterion can be expressed in terms of phase velocities of neutral waves. Thus, for a given integer n, *the cross-stream decay of pressure fluctuations is algebraic only if neutral waves have supersonic phase velocities* (*relative to the ambient sound speed*).

In the following section, we show that a similar transition holds in the cross-stream direction when the phase velocity of the neutral axisymmetric ($n = 0$) instability wave is subsonic, in a way which is given by a turning-point problem. It is shown that the feature of the near-field is completely compatible with the asymptotic expansion (2.9) valid in the intermediate region and with the far-field region.

2.3. A turning-point problem

In this section, we study the cross-stream behaviour of local plane waves in the annular region $|\lambda| = \mathrm{O}(1)$, $|\lambda| \neq \mathrm{o}(1)$ centered at the point k_0 at the limit $\omega_i \to 0$. We show that the analysis of Eq. (2.6) in the complex plane permits to identify a turning-point problem. The following statements will be proven:

(a) *for $|k_r| > k_0$ and $k_i = 0$ the cross-stream behaviour of axisymmetric instability waves ($n = 0$) is dispersive in a neighbourhood of the jet if λ is small enough. For $n \neq 0$ or $r > 1/(2\lambda)$, the cross-stream behaviour is dissipative.*

(b) *for $|k_r| < k_0$ and $k_i = 0$ the cross-stream behaviour of axisymmetric instability waves is dispersive. If $n \neq 0$, the cross-stream behaviour is dispersive only outside a neighbourhood of the jet if λ is small enough.*

The proof of statements (a) and (b) can be carried out by the same way. Let us first $z = \mathrm{i}\lambda r$ be a new independent variable and reduce Eq. (2.6) to its normal form by writing $\hat{u}_{n0} = u_{n0} z^{1/2}$ in (2.6) and $p = \pm n - 1/2$, where u_{n0} is a suitably normalized function identically equals to the n-th-order Hankel function of the first kind outside the jet. Then, we have

$$\frac{\partial^2 \hat{u}_{n0}}{\partial z^2} + Q(z)\hat{u}_{n0} = 0, \qquad Q(z) = 1 - \frac{p(p+1)}{z^2}, \tag{2.10}$$

where the function Q is regular in the whole of the domain $\mathcal{E}$. Eq. (2.10) defines a turning-point problem (see Bender and Orszag [16]). The turning points are points where Q vanishes, they are given by

$$r_0 = \pm \frac{\sqrt{p(p+1)}}{\mathrm{i}\lambda(k)}, \tag{2.11}$$

where λ is defined by (2.7) on its Riemann surface. It follows that r_0 is real only if $|k| > k_0$ and $k_i = 0$ for $n = 0$, or $|k| < k_0$ and $k_i = 0$ for $n \neq 0$. Then by Eq. (2.10) the cross-stream behaviour of neutral axisymmetric instability waves is dispersive up to $r_0 = 1/(2\lambda)$ if $|k| > k_0$. Since (2.10) is valid for $|\lambda| = \mathrm{O}(\varepsilon^{\alpha})$, $|\lambda| \neq \mathrm{o}(\varepsilon^{\alpha})$, the turning-point gives the radial position of the dispersive–dissipative transition in the intermediate domain for $n = 0$. This result is completely compatible with the asymptotic expansion (2.9): the rapidly varying component of (2.9) decays exponentially (dissipates) away from the jet as long as $\varepsilon^{\alpha} = \mathrm{o}(|\lambda|)$. For $|k| < k_0$ and $k_i = 0$, the turning point lies in the complex r-plane and the cross-stream behaviour becomes dispersive. As indicated by (2.9) the solution is wavelike with very small and slowly changing wavelengths and slowly varying amplitude as function of r_α. The case $n \neq 0$ and $|k| > k_0$ leads to a dissipative behaviour. Therefore, the statement (a) is proved.

The proof of statement (b) is very similar. Let us consider the case $|k| < k_0$, $k_i = 0$ and $n \neq 0$, for which r_0 is real. Hence, turning to the definition of λ on its Riemann surface, we can assert that the cross-stream behaviour of instability waves is dispersive for $r > r_0$ and dissipative in a subdomain $r < r_0$ outside the jet. By the definition of r_0, the length of this subdomain is an increasing function of n and consequently, the cross-stream decay of instability waves is exponential for $r \leqslant \mathrm{O}(n)$. Thus, the absolute value of u_{n0} is negligibly small for large n and $r > r_0$. For axisymmetric instability waves, the function Q is complex for all r outside the jet and the cross-stream behaviour is dispersive.

Recall that the inequalities $|k_r| > k_0$ and $|k_r| < k_0$ must be considered in the sense of asymptotic analysis, that is, $|\lambda| = \mathrm{O}(1)$ and $|\lambda| \neq \mathrm{o}(1)$. If $|\lambda| = \mathrm{o}(\varepsilon^{\alpha})$ one might consider trying to apply perturbation methods to find the asymptotic structures. The leading-order of Eq. (2.4) reduces to the transverse part of the Laplace operator:

$$L_0 u_{n0} = \left\{ \frac{\partial^2}{\partial r_\alpha^2} + \frac{1}{r_\alpha}\frac{\partial}{\partial r_\alpha} - \frac{n^2}{r_\alpha^2} \right\} u_{n0} = 0. \tag{2.12}$$

The form of the solution no longer depends on the phase velocity but is stronger affected by the azimuthal wavenumber n. For axisymmetric instability waves, the general solution corresponds to the asymptotic form of the Hankel function for small argument, that is $u_{00}(z) \sim \ln z$, with $z = \mathrm{i}\lambda r_\alpha/\varepsilon^{\alpha}$. This solution obviously breaks down at large distances from the jet. From matching arguments we deduce that the turning-point becomes larger than $\mathrm{O}(1)$ and has no physical sense. For non-axisymmetric instability waves, the solution which matches the Hankel function is given by $u_{n0}(z) \sim 1/z^n$, for $n > 0$. Let us note that the branch points go to zero as $M \to 0$ for finite values of ω and, consequently, the length of the vertical strip between the two branch points, representing waves with supersonic convective Mach number, decreases until zero. Hence only axisymmetric pressure fluctuations may exhibit a dispersive behaviour in the vicinity of low-speed jets.

2.4. *Branch points and far-field*

Since outside the jet, the linear operator is homogeneous in space variable x, it is possible to make use of the Fourier–Laplace transform. As solutions of the linear wave equations may be expressed in terms of Hankel functions, the pressure field may be simplified to

$$p_n(x, r, \omega) \sim \int_{-\infty}^{+\infty} G(K; \omega) H_n^{(1)}\big(\mathrm{i}\lambda(K; \omega) r\big) \exp(\mathrm{i}Kx)\, \mathrm{d}K, \tag{2.13}$$

where the algebraic function is identified to its first analytic branch $\lambda_1(K; \omega)$ for the boundary condition to be satisfied. Following Tam and Morris [15], the function G is the Fourier transform of the leading-order of the WKBJ approximation

$$G(K; \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A_0(X; \omega) \exp\left(\frac{\mathrm{i}}{\varepsilon} \int_0^X k(s; \omega)\, \mathrm{d}s - \mathrm{i}Kx \right) \mathrm{d}x. \tag{2.14}$$

In a spherical coordinate system (η, χ, θ) centred at the nozzle exit of the jet with the polar axis aligned in the direction of the flow, the far-field can be evaluated asymptotically (2.13) for large R by applying the method of stationary-phase. Following standard arguments and considering the asymptotic form (2.9) of the Hankel function, one obtains that the stationary phase point K^* is defined by $K^* = \pm k_0 \cos\chi$, which implies $|K^*| < k_0$. Thus, one finds that the stationary phase point of Eq. (2.14) exists only if the phase in the integrand satisfies

$$\frac{\partial}{\partial X}\left(\int_0^X k(s; \omega)\, \mathrm{d}s \mp k_0(\omega)\cos\chi \right) = 0, \tag{2.15}$$

for some frequencies ω, that is, if the (single) stationary point X^* is such that

$$k(X^*; \omega) = \pm k_0(\omega)\cos\chi. \tag{2.16}$$

Since the right term of (2.16) is bounded by k_0, this condition is necessarily satisfied if the spatial branch meets a branch cut. In other terms, the exponential-algebraic transition leads to acoustic radiation in the far-field. Physically, the radiated sound field is no longer an insignificant part of the total phenomenon. The disturbances associated with the flow-instability process now extend from the jet all the way to the far field. With the notations of Section 2.2, the direction is given by $\cos\chi = k_r^*/k_0$ and depends on ω. Note that the branch cuts of the local problem are induced by the boundary conditions, through the matching process given by (2.14).

Since the phase of λ on the first sheet of the Riemann surface satisfies Eq. (2.8), the wavenumber of the asymptotic evaluation of (2.13) is obviously the branch point k_0. One obtains finally

$$p_n(\eta, \chi; \omega) \sim \frac{2}{\eta} G\big(k_0(\omega)\cos\chi; \omega\big)\exp\big(\mathrm{i}k_0(\omega)\eta\big). \tag{2.17}$$

The next section deals briefly with the numerical formulation for performing calculations of instability waves. Using the structure of solutions on the Riemann surface, the computational domain has been reduced to the vicinity of the jet. Here, a new method has been used to discretized the operators. The mathematical and computational steps needed to calculate the spatial branches will not be fully explained. A more complete explanation of the numerical procedure will be published elsewhere.

3. Numerical formulation

The approximate solution requires two steps. First the operator associated with the linearized equations is converted to a matrix. In the present method the discretized system is obtained by a spectral collocation discretization through a multiple domain technique. Such a method can be applied with no modification when the viscosity effects are taken into account. In the second step the resulting nonlinear eigenvalue problem (nonlinear in k) is solved at every axial location X by using the linear companion matrix method when possible.

3.1. Spectral discretization

The differential equations are reduced to linear algebraic equations using a spectral collocation method (see Khorrami et al. [12], Malik [13]). One drawback to these techniques has been the requirement that a complicated physical domain must map onto a simple computational domain for discretization. Let $r = f(t)$ denote a mapping of the computational coordinate t into the natural coordinate r. The convergence properties of the approximation to $\phi(r)$ can be determined from the behaviour of the function $\phi(f(t))$. The function f must be infinitely differentiable if the high-order accuracy and exponential convergence rates associated with spectral methods are to be preserved. Additionally, even smooth stretching transformations can decrease the accuracy of a spectral method, if the stretching is severe. The latter restriction is overcome in the present method by splitting the physical domain into two domains which may be denoted as domain 1 and domain 2. The only point which belongs to both the domains is the inflection point of the streamwise component of the mean velocity profile. Each domain preserves the advantages of spectral collocation and allows the ratio of the mesh spacings between regions to be several orders of magnitude larger than allowable in a single domain.

Since the problem of interest here is nonperiodic, Chebyshev polynomials are suitable basis functions. The standard collocation points are the Gauss–Lobatto points for every domain. The derivative matrices are constructed for both the domains as described for a single domain (see Malik [13]). The dimension of the unknown vector depend on the number of dependent variables. In most of computations of this paper, the compressible version of the Rayleigh equation is used, and so the vector has only one component: the pressure. However for reasons which will appear in the following section, it is not possible to extent this approach to small frequencies. Indeed, the linearized, viscous, compressible equations may be considered, with a large Reynolds number in the spirit of Tam and Chen [17]. Then, the vector is composed of velocity components, pressure and density. For both cases, interface conditions are obtained by requiring continuity of components and their derivatives.

3.2. Inviscid approximation

To simplify the necessary calculations one can use the inviscid stability theory since the mean flow is dynamically unstable even in the absence of viscosity. Such a problem requires solving a differential equation with singularities on or near the computational domain. These singularities, usually called “critical points” are particularly severe for Chebyshev methods since these global expansions algorithms are very sensitive to the analytic properties of the solution.

As has been discussed by many investigators, the solution for inviscid damped waves is obtained by analytic continuation of that of the unstable wave in the complex r-plane. By making a change of coordinates, one can solve the problem along a path in the complex plane that makes a wide detour around the singularity. The new integration contour must pass through both boundary points and should be as smooth as possible, as indicated by Boyd [18]. Unfortunately, the stretching transformation leads to a new distribution of nodes along the complex contour and particularly around the critical point. Following anew the multi-domain spectral collocation method, the domain which contains the critical point is divided into two domains when necessary. In some cases, the critical point is too close to the origin for the complex mapping method to be applied. This justifies, a posteriori, taking into account viscosity in our calculations.

Since the WKBJ approximation is valid only inside and in the immediate vicinity of the jet, the boundary conditions for the discretized system may depend on k. For exponential decay, one can find a region far enough from the inflection point where the pressure fluctuation is negligibly small. Therefore in the numerical computation it is taken to be zero. Such a boundary condition fails for algebraic cross-stream decay. In this case, the asymptotic expansions of the n-th-order Hankel functions show that the necessary length of the integration domain would become too large for a reasonable number of points. Instead the pressure fluctuation is required to match the n-th-order Hankel function of the first kind $H_n^{(1)}$ outside the jet, that is, complex pairs (k,ω) are determined by numerical integration of the Rayleigh equation together with the boundary conditions

$$p_{n0}(0)=0 \quad \text{and} \quad p_{n0}(r_\infty)=H_n^{(1)}(\mathrm{i}\lambda r_\infty), \tag{3.1}$$

where r_∞ is the upper bound of the integration domain. The choice of the regular branch for λ depends on the sheet of the Riemann surface under consideration. It must be noted that now the discretized problem can no longer be written as a polynomial in the parameter k.

3.3. *Global or local method?*

There are two classes of numerical methods that can be used for computing the eigenvalues: global and local methods. For the global methods a generalized eigenvalue problem is set up and the eigenvalues are obtained by using standard algorithms. When the discretized operator is a polynomial in the parameter k, several methods exist for determining the set of eigenvalues without an initial estimate. Following, the linear companion matrix method (see Bridges and Morris [19]), the linear differential operator is converted to a generalized eigenvalue problem. But this method does not apply if the boundary conditions are not polynomials in the parameter k. In such a case, we use a local method. At each value of ω, the wavenumber is performed by searching for the zero of the function

$$p'_{n0}(r_\infty)-\mathrm{i}\lambda H_n^{(1)'}(\mathrm{i}\lambda r_\infty)=0. \tag{3.2}$$

In a local method, a guess for the eigenvalue is required. Only the eigenvalue which happens to lie in the neighbourhood of the guessed value is computed using iterative techniques such as Newton's method to solve (3.2). Thereafter, the eigenvalue solution at the previous axial location can be used as the initial guess for the eigenvalue at the next axial location. Extrapolating the first guessed eigenvalue at the next axial location from previous values often speeds up the convergence as long as the mean flow profiles are slowly changing.

4. Numerical results for cold supersonic jets

4.1. *Mean-flow profile*

Before numerical calculations can be performed, it is necessary to provide a description of the mean velocity and density profiles in the jet. The mean velocity profiles are taken from experimental measurements of perfectly expanded supersonic jets obtained by Lau et al. [3] whose data were fitted by an error function profile and Troutt and McLaughlin [2] who fitted data by a half-Gaussian profile.

It is known that a supersonic jet can be divided into three regions, i.e. the core, the transition and the fully developed regions according to the characteristics of the mean velocity profile. Here, we restrict the analysis to the core region where the mean velocity may be approximated by a half-Gaussian profile. In the core region, close to the nozzle exit, the center part of the jet has a uniform velocity which is equal to the fully expanded jet velocity. Surrounding the core, there is a mixing layer which broadens in the downstream direction. In the present work the mean velocity is approximated by

$$\bar{u}(r;X)=\begin{cases}1 & \big(r<h(X)\big),\\ \exp\left(-\ln 2\left(\dfrac{r-h(X)}{b(X)}\right)^2\right) & \big(r\geqslant h(X)\big),\end{cases} \tag{4.1}$$

where h is the radius of the potential core and b is the velocity half-width of the annular mixing region. The density is related to the mean axial velocity using a Crocco relationship,

$$\bar{\rho} = \left(\bar{u} + \frac{T_\infty}{T_0}(1-\bar{u}) + \frac{\gamma-1}{2}\bar{u}(1-\bar{u})M^2\right)^{-1}, \tag{4.2}$$

where T_∞ and T_0 denote the ambient temperature and the jet exit temperature, respectively. Here, we only consider cold jets, that is, T_∞/T_0 only depends on the Mach number. On substituting for $\bar{u}$ and $\bar{\rho}$ Eqs. (4.1) and (4.2) into the momentum integral equation, an algebraic relationship can be found between h and b in the annular mixing region. Then, the axial development of the jet is completely defined by the axial variation of the jet half-width b.

Extensive jet instability calculations over a significant range of Strouhal number have been carried out. For most cases, it is found that the position of neutral instability is located in the core region of the jet. For the purpose of determining the intersection point k^* it is, therefore, not necessary to consider the transition and the developed regions of the jet. The length of the uniform core is obtained by using the modified formulae given by Tam et al. [20]. In the literature $\mathrm{d}b/\mathrm{d}x$ is referred to as the spreading rate σ of the mixing layer. For the Gaussian mean velocity profile, we have the relation $\mathrm{d}b/\mathrm{d}x = \varepsilon = 1.2658/\sigma$. The variation of σ as a function of the jet exit Mach number, M, has been tabulated by Birch and Eggers [21]. The correlations are given by

$$\begin{aligned} &\sigma = 10.7/\left(1.0 - 0.1163M^2\right), \quad M < 2.0,\\ &\sigma = 19.4\sqrt{M - 0.9418}, \quad M > 2.0. \end{aligned} \tag{4.3}$$

As an example, for $M = 1.5$, we find $\mathrm{d}b/\mathrm{d}x = \varepsilon = 0.088$. Note also that for supersonic cold jets, experimental measurements have shown that $\mathrm{d}b/\mathrm{d}x \leqslant 0.10$.

4.2. General methodology and numerical results

In this section numerical results about local pairs (k^*, ω^*) for which $k(X;\omega^*) \cap K(\omega^*)$ is non-empty, will be presented for Kelvin–Helmholtz modes with small azimuthal wavenumbers $(0 \leqslant n \leqslant 3)$. These pairs were determined by numerical integration of the linearized equations of motion, together with boundary conditions outside the jet. At each value of the axial location X, the phase velocity of a neutral wave was located by searching for the zero of a growth rate $-k_i$ for positive ω. As the location is increased from the origin to the end of the core region, the spatial branches move into the upper half of the complex k-plane and, consequently, points of zero growth rates describe curves along the real axis $k_r > 0$. As discussed in the Section 2.2, some of these curves may lie in the vertival strip $|k_r| < k_0$, corresponding to supersonic phase velocities. According to the definition of the branch cut K, the segments of these curves for which $|k_r| < k_0$ and their images in the ω plane give complex pairs (ω^*, k^*) as functions of the axial location, for which the cross-stream decay of pressure fluctuations is algebraic. The cross-stream wavenumber is then given by the function $\lambda(k^*;\omega^*)$ and depends on n through (ω^*, k^*). The above reasoning presents the advantage of uniquely determining which spatial branches are pertinent to acoustic radiation outside the jet. The numerical procedure to determine the spatial branches involves the definition of the function λ on its Riemann surface. Subsequently, spatial branches are analytically continued into the second Riemann sheet for damped waves $(k_i > 0)$, if necessary. This is permissible because boundedness of solution at large r is not a requirement of the near-field solution. The associated parametric dependence of the real and imaginary parts k_r and k_i on ω and X is displayed in Fig. 3, for a cold jet of Mach number $M = 1.7$ and for axisymmetric $(n = 0)$ and helical $(n = 1)$ Kelvin–Helmholtz modes. Observe that for high frequencies the spatial branches have the same behaviour, whereas for low frequencies the phase velocity depends strongly on the azimuthal wavenumber. A consequence is that for $x = 10$, there is no intersection point k^* for the helical mode. But it still exists for the axisymmetric mode, as indicated by the analytic continuation into the second Riemann sheet, in Fig. 3.

For $M = 1.7$, Fig. 4 shows branches described by points of zero growth rates, in the limit process $k_i \to 0$, $k_i > 0$, on a multisheeted plane with two connected Riemann sheets. Each curve can be divided into two segments, according to the cross-stream structure of the pressure fluctuations. The first branch (sheet 1 in Fig. 4) is associated with exponential decays, satisfying the boundedness condition as $r \to \infty$. The second branch (sheet 2) lies along the branch cut, so that, the cross-stream structure of pressure fluctuations is algebraic at the limit $k_i \to 0$. Application of the criterion for exponential-algebraic transition reveals that only small azimuthal wavenumbers contribute to acoustic radiation in the far field. For instance, the line associated with $n = 3$ passes along a single sheet of the Riemann surface. This simple topological configuration leads to an exponential decay in the cross-stream direction for all X. It follows that the dominant contribution of instability waves to acoustic radiation in the near field $(r_\alpha = \mathrm{O}(1))$ is given by segments of the real ω-axis, depending on small azimuthal wavenumbers and control parameters. The mapping of these segments through $\omega^* = \omega^*(X)$ into the physical domain shows locations X for which it is possible to find a frequency ω^* such that $k(X;\omega^*) \cap K(\omega^*)$ is nonempty, that is, the cross-stream decay of disturbances is algebraic. For the axisymmetric Kelvin–Helmholtz mode, Fig. 5 shows that the lower bound of Ω^* is displaced into the fully

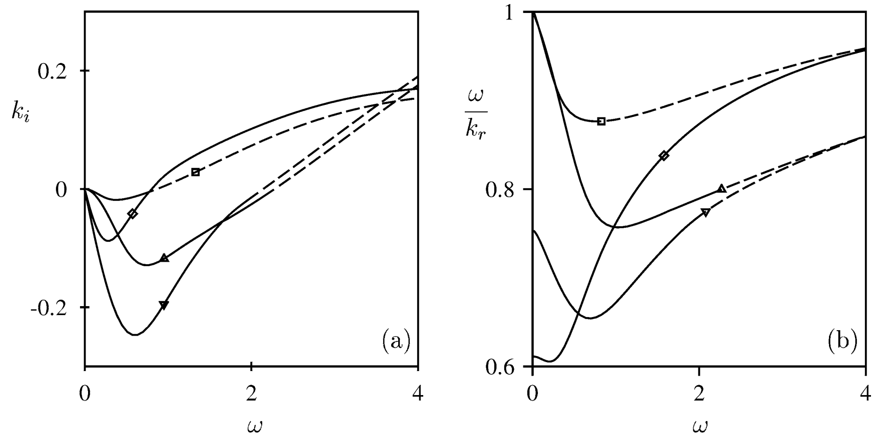

Fig. 3. Evolution of spatial branches associated with the axisymmetric ($n = 0$) and the helical ($n = 1$) Kelvin–Helmholtz modes as the axial location is increased in the core region, for a cold supersonic jet of Mach number 1.7: (a) imaginary part k_i of k versus real frequency ω; (b) phase velocity ω/k_r versus ω. $\triangledown$, $x = 5$, $n = 1$; $\triangle$, $x = 5$, $n = 0$; $\square$, $x = 10$, $n = 0$; $\diamond$, $x = 10$, $n = 1$. Dashed line: analytic continuation into the second Riemann sheet.

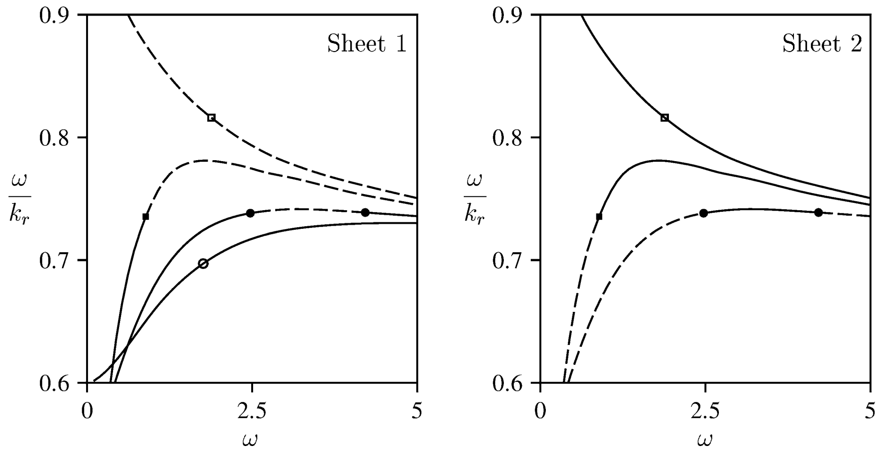

Fig. 4. Phase velocity of weakly damped instability waves ($k_i \to 0$, $k_i > 0$) as a function of the frequency when the axial location is increased from the origin to the end of the core region. Cold jet of Mach number $M = 1.7$. $\square$, $n = 0$; $\blacksquare$, $n = 1$; $\bullet$, $n = 2$; $\circ$, $n = 3$; dashed line, analytic continuation into the other Riemann sheet. The solid lines on the sheet 2 give rise to algebraic decay in the cross-stream direction.

developed region, as $M \to 1$, $M > 1$, where the centreline mean velocity of the jet is a decreasing function of X. In this last case, numerical computations of spatial branches may be continued in the developed jet region in order to have this upper bound. But, the present approach fails when the order of magnitude of the frequency becomes comparable with that of the slow space variable $X = \varepsilon x$. Note also that the upper bound is not located in the core region. For non-axisymmetric modes, Fig. 6 shows that the lengths of these segments are zero for low supersonic jets (typically $M < 1.6$ in Fig. 6). Thus, we conclude that for low supersonic jets, only axisymmetric disturbances may generate acoustic radiation in the far-field; the contributions from non-zero azimuthal wavenumbers represent a near-field of hydrodynamic type. For high supersonic jets, the entire history of spatial evolutions of instability waves need to be taken into account in determining the contribution of each azimuthal wavenumber and the physical regions concerned by the radiation of sound.

For $\alpha = 1$ ($r_\alpha \sim X$), the cross-stream decay of fluctuations is not simply given by a Hankel function. The pressure field may be represented by a Fourier integral, as shown in Section 2.4, giving both the hydrodynamic components as well as the radiated sound. Note also that this result fails in small neighbourhoods of branch points, where a complete asymptotic analysis must be developed by considering λ as a small parameter.

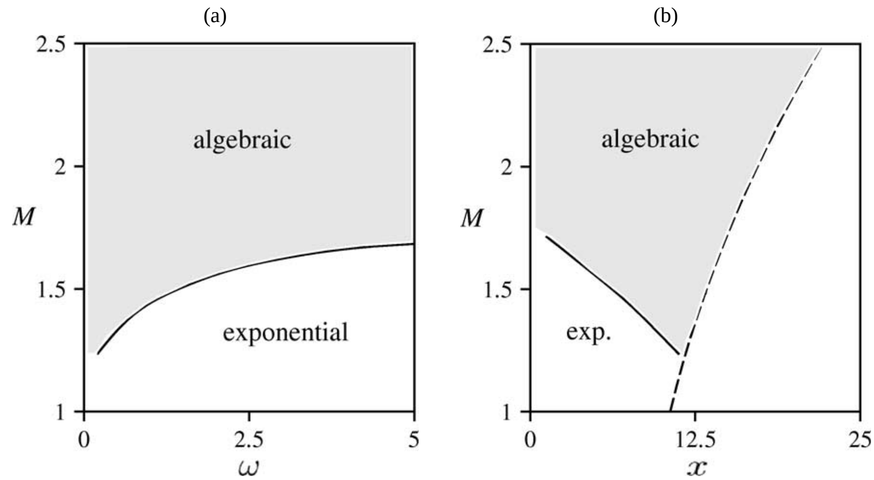

Fig. 5. Domain of exponential-algebraic transition in M–ω (a) and M–x (b) planes for $n = 0$. The dashed line in (b) gives the end of the core region. For each point located above the curve, into the grey domain, the cross-stream decay is algebraic at a location corresponding to the stationary phase point of the pressure far-field.

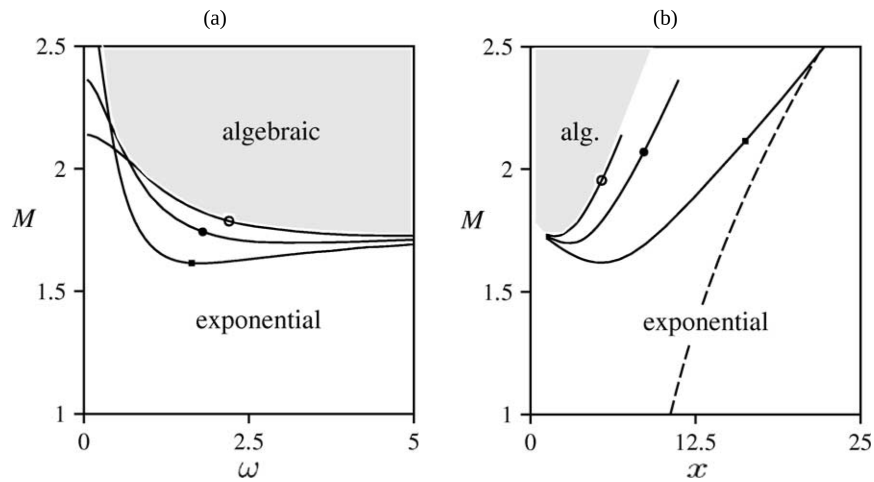

Fig. 6. Domain of exponential-algebraic transition in M–ω (a) and M–x (b) planes for non-axisymmetric modes. ■, $n = 1$; •, $n = 2$; ◦, $n = 3$; the dashed line in (b) gives the end of the core region.

5. Conclusion

We have examined the near-field pressure decay generated by instability waves in axisymmetric jets as a source of noise. Exponential decay of pressure fluctuations is found to change to algebraic around locations X which depend on the azimuthal properties of the local plane waves. This transition is a condition for acoustic radiation to the far-field, where the wavelength is given by the branch point, and so decreases with the Mach number.

It has been shown that for low supersonic jets ($M \to 1$, $M > 1$) the location of the lower bound of the radiating region associated with the axisymmetric Kelvin–Helmholtz mode moves inside the developed jet region. It follows that the cross-stream wavenumber, goes to zero, so the wavelength may become larger than the envelope scale. According to numerical results, other azimuthal modes no longer generate sound in the near field.

Finally we note that the generalization of these concepts to subsonic jets is an open question (see Cooper and Crighton [22] and Crighton and Huerre [23] for a discussion of superdirective sources in low-speed jets). For $M = 0$ and $\omega_i = 0$, the branch cut K is identical with the imaginary axis and one can use the Hilbert transform of the perturbation to reduce the analysis to positive wavenumbers (for instance, see Huerre and Monkewitz [24]).

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Appendix. Equation outside the jet

In this appendix Eq. (2.4) to be solved outside the jet is obtained from equations of motion. On factoring out the exponential dependence on θ and t the governing equations for the spatial part $\phi_n = (\mathbf{u}_n, p_n)$ of the solution are obtained from (2.1a) and (2.1b). With respect to a cylindrical coordinate system (x, r, θ) centred at the nozzle exit and the corresponding velocity components (u_n, v_n, w_n), the governing equations are

$$-\mathrm{i}\omega u_n + \frac{\varepsilon\bar{v}_\infty}{r}\frac{\partial u_n}{\partial r} = -\frac{1}{\bar{\rho}_\infty}\frac{\partial p_n}{\partial x}, \tag{A.1a}$$

$$-\mathrm{i}\omega v_n + \frac{\varepsilon\bar{v}_\infty}{r}\frac{\partial v_n}{\partial r} - \frac{\varepsilon\bar{v}_\infty}{r^2}v_n = -\frac{1}{\bar{\rho}_\infty}\frac{\partial p_n}{\partial r}, \tag{A.1b}$$

$$-\mathrm{i}\omega w_n + \frac{\varepsilon\bar{v}_\infty}{r}\frac{\partial w_n}{\partial r} + \frac{\varepsilon\bar{v}_\infty}{r^2}w_n = -\frac{1}{\bar{\rho}_\infty r}\mathrm{i}np_n, \tag{A.1c}$$

$$-\mathrm{i}\omega p_n + \frac{\varepsilon\bar{v}_\infty}{r}\frac{\partial p_n}{\partial r} + \frac{1}{M^2}\left(\frac{\partial v_n}{\partial r} + \frac{v_n}{r} + \frac{\mathrm{i}nw_n}{r} + \frac{\partial u_n}{\partial x}\right) = 0. \tag{A.1d}$$

Substitution of (A.1a), (A.1b) and (A.1c) into (A.1d) it is straightforward to find that the equation to be solved can be cast into the form

$$\begin{aligned}
&\delta_R^2\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{n^2}{R^2}\right)u_n + \left(k_0^2 + \delta_X^2\frac{\partial^2}{\partial X^2}\right)u_n \\
&\quad = \left(\delta_R^4\bar{\rho}_\infty\left(\frac{\varepsilon\bar{v}_\infty M}{R}\right)^2\left(\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R}\right) - 2\delta_R^2\frac{\varepsilon\bar{v}_\infty M^2}{R}\mathrm{i}\omega\bar{\rho}_\infty\frac{\partial}{\partial R}\right)u_n,
\end{aligned} \tag{A.2}$$

where the outer variables R and X are defined by $R = \delta_R(\varepsilon)r$ and $X = \delta_X(\varepsilon)x$. The order functions δ_R and δ_X are real, continuous functions of ε in a neighbourhood of the origin and such that $\delta_X \sim \delta_R$ far away from the center of the jet.

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